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Upper-bound problem for a rotating system

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Recent work (Hunter & Riahi 1975) on nonlinear convection in a rotating fluid is extended to a multi-modal regime. The schematic multi-boundary-layer method of Busse (1969) and the upper-bound technique of Howard (1963) are used to obtain upper bounds on the Nusselt number N. It is shown that there are infinitely many modes in the range $Ta \ll R^{\frac{3}{2}}$, where Ta is the Taylor number and R is the Rayleigh number, and different types of mode optimize N in different regions of the parameter space (R, Ta). While the optimal N is independent of Ta for $Ta \ll R$, it is found that it increases with Ta in $R \ll Ta \ll (R \log R)^{\frac{4}{3}}$ and decreases as Ta increases in $(R \log R)^{\frac{4}{3}} \ll Ta \ll R^{\frac{3}{2}}$, and that the functional dependence of the optimal N on R and Ta is continuous (within a logarithmic term) throughout the region of R, Ta space.

1. Introduction

We consider the effect of rotation on convection between two rigid horizontal boundaries at large Rayleigh number. A further application of the upper-bound calculation extends the recent work of Hunter & Riahi (1975, henceforth referred to as I) to the multi-modal case.

The multi-boundary-layer method was first used by Busse (1969). In improving the upper bound on the heat flux, Busse considered a sequence of different boundary layers by adjusting the horizontal scale from its interior value to its boundary value. The thickness of each boundary layer was supposed to be large in comparison with the thickness of the following layer, and the convecting component of the heat flux was supposed to be approximately equal to the total heat flux in all but the last of the boundary layers, where it was of the order of the total heat flux. Later on, Chan (1971, henceforth referred to as II) used Busse's technique to study turbulent convection at infinite Prandtl number, and obtained the preferred upper bound on the heat transport. Since then, this technique has been used by Busse & Joseph (1972), Gupta & Joseph (1973) and Chan (1974; henceforth referred to as III) to study nonlinear convection. In all such studies, a schematic structure for all the modes was considered. Also, it was assumed that higher modes have shorter length scales and that coupling among the different modes occurs only between the $(n+1)$th and the nth mode in the nth boundary layer.

A single-mode analysis of the present problem is performed in I. It is found that in the range $R \ll Ta \ll (R \log R)^{\frac{4}{3}}$ there is a mode, referred to as the E-mode, which has a length scale of order $Ta^{-\frac{1}{6}}$. The boundary-layer structure for the E-mode consists of a non-uniform interior, an Ekman layer of thickness $(4/Ta)^{\frac{1}{4}}$ and an inner layer thinner than the Ekman layer. The thickness of the Ekman layer is found to be fixed and independent of the horizontal cell structure. Such a particular structure makes it impossible to have an additional E-mode which has the schematic structure. In this paper, we find that, in order to preserve the schematic multi-boundary-layer structure in the above range of Ta, there must be infinitely many modes, called R-modes, which have the same structure as the ones in a weakly rotating system (II) and finitely many modes, called T-modes, which have the same structure as the ones in a strongly rotating system (III); furthermore, it is shown that T-modes have no significant effect on the optimization of N. Thus only the effect of R-modes on maximizing N is considered. Whether there is a different kind of multi-boundary-layer structure having many E-modes is an unanswered question, but it is clear that, if there is such a non-schematic structure, it should have the property that all of its E-modes are coupled throughout the Ekman layer. The cellular structure [I, equation (2.16)] for the solutions and the possible physical interpretation of the modes as small scales of motion suggest that such a structure is physically improbable for the present problem.

2. Multi-modal regime

First we state the problem. The reader is referred to I and Riahi (1974) for details of the subject and the derivation of the equations.

We consider an infinite horizontal layer of fluid of depth d bounded above and below by two rigid, perfectly conducting planes maintained at temperatures T_0 and $T_0 + \Delta T$ respectively. The fluid is rotating about the vertical with angular velocity Ω. Under the usual Boussinesq approximation and the assumption of infinite Prandtl number, the basic equations can be determined [I, equations (2.8), (2.9) and (2.14)]. We seek the maximum value of N subject to the constraints provided by these basic equations. From the Euler equations for the variational problem, we obtain our governing equations [I, equations (2.19) to (2.21), where the subscript n is added to each variable]. These governing equations are then used to obtain multi-modal solutions. For further details on the mathematical analysis of the problem, the reader is referred to II and III.

2.1. *The case $Ta \ll R$*

As in the single-mode case (I), rotational effects are unimportant so long as $Ta \ll R$. There are no significant differences between the solutions for this case and those for the non-rotating case (II). We refer to each of the modes for this case as an R-mode. Each R-mode (of wavenumber α_n) has three regions: the interior, the intermediate layer and the inner layer. The interior of each mode coincides with the inner layer of the previous mode. Coupling among the different modes occurs only between the nth and the $(n-1)$th mode in the $(n-1)$th boundary layer. It is assumed that

$$Ta \ll \alpha_1^4, \tag{1}$$

$$\delta_S \ll \ldots \ll \delta_n \ll 1/\alpha_n \ll \delta_{n-1} \ll \ldots \ll \delta_1 \ll 1/\alpha_1 \quad \text{as} \quad \alpha_n \to \infty, \tag{2}$$

where $1/\alpha_n$ and δ_n are the thicknesses of the intermediate and inner layers of the nth mode respectively, and S is the total number of R-modes. Using a formal multi-boundary-layer technique (II, § 5), we find that there can be infinitely many R-modes. N is independent of Ta and increases with R. Chan (II) used a less formal approach to find the multi-modal solutions of the upper-bounding problem with no rotation and obtained the same qualitative results. However, in his result for N, he found a different proportionality constant. This is partly due to his ignoring the contributions of the term $R^{-\frac{1}{3}\times 10^{-s}}$ in the expression for N, and consequently the coefficient $e^{-\frac{8}{9}}$ does not appear in his equation (95) for the heat flux.

2.2. *The case* $R \ll Ta \ll (R \log R)^{\frac{4}{3}}$

It is found in I that for this case there is a mode, referred to as the E-mode, which has a non-uniform interior (a combination of a uniform interior and a layer of thickness $Ta^{\frac{1}{2}}/\alpha_E^3$ as $\alpha_E^6 \to Ta$), an Ekman layer of thickness $(4/Ta)^{\frac{1}{4}}$ and an inner layer thinner than the Ekman layer. Also, it is shown that this inner layer has the same structure as the inner layer of the R-modes and has thickness

$$\delta_E \propto (Ta^{\frac{1}{2}} R \log Ta)^{-\frac{1}{3}}.$$

Under what conditions can the number of modes be increased? From the information on the E-mode, we see that the Ekman layer is thicker than the inner layer, and that its thickness is fixed and independent of the horizontal cell structure when

$$R \ll Ta \ll (R \log R)^{\frac{4}{3}}. \tag{3}$$

This clearly shows that the system has not more than one E-mode with the usual multi-boundary-layer structure. By assumption (1), we have the usual multi-boundary-layer structure and α_n modes (in addition to the E-mode) exist which have the same structure as the R-modes. Chan's analysis for free boundaries (III) shows that there are infinitely many modes, called T-modes, when

$$R^{\frac{4}{3}} \ll Ta \ll R^{\frac{3}{2}} \tag{4}$$

and finitely many when

$$R \ll Ta \ll R^{\frac{4}{3}}, \tag{5}$$

where in the latter range the number of modes increases as Ta increases. The structure of the T-modes is found to be independent of the boundary conditions so long as

$$(R \log R)^{\frac{4}{3}} \ll Ta \ll R^{\frac{3}{2}} \quad \text{(I, III)}. \tag{6}$$

From I, for the case of rigid boundaries there is an Ekman layer outside the T-modes in the range (6). The structure of the interior of the E-mode is basically the same as that of a T-mode, and it turns out that we also have finitely many T-modes in the range (5). Each of these T-modes satisfies the relation

$$\alpha_n^4 \ll Ta \ll \alpha_n^6. \tag{7}$$

The analysis for the range studied in §2.3 (though omitted in this paper) shows that assumption (7) is basic for the existence of a T-mode. It can be shown from (7), (18) and (19) that none of the T-modes can have a significant effect on the optimization of N in the range (3).

To determine the total number of T-modes in the range (3), we use the condition $Ta^{-\frac{1}{4}} \ll \delta_n$ and equations (18) and (19) in §2.3. We find that, for $R \ll Ta \ll R^{\frac{6}{5}}$, there is no T-mode. For $R^{\frac{6}{5}} \ll Ta \ll R^{\frac{9}{7}}$, there is only one T-mode. In general, if we ignore the logarithmic terms, a total of l T-modes exist when

$$R^p \ll Ta \ll R^q, \quad \text{where} \quad p = \frac{12(1-2^{-l})}{9-2^{3-l}}, \quad q = \frac{12(1-2^{-l-1})}{9-2^{2-l}} \tag{8}$$

for a given R. We note that l increases with Ta in the range (5) and that, as $Ta \to R^{\frac{4}{3}}$, $l \to \infty$ and the lower and upper parts of the inequality (8) merge. It can be shown easily also that, for a given Ta, l is determined uniquely from (8).

To optimize N, we use the usual schematic multi-boundary-layer method and assume there are many R-modes in addition to the E-mode. We omit the analysis here since it is similar to that in II. We obtain the following important results:

$$\alpha_n = b_n R^{\frac{1}{3}(1-10^{-n})} \prod_{K=1}^{n-1} (\log 1/g_K)^{\frac{1}{2}\times 10^{K-n}} (Ta^{\frac{1}{2}} \log Ta)^{\frac{1}{2}\times 10^{-n}}, \tag{9}$$

$$g_n^{10^n} (\log 1/g_n)^{2\times 10^{n-1}} \prod_{K=1}^{n-1} (\log 1/g_K)^{-3\times 10^{K-1}} = R^{-\frac{1}{5}} (Ta^{\frac{1}{2}} \log Ta)^{\frac{3}{10}}, \tag{10}$$

$$N = K_S (Ta^{\frac{1}{2}} \log Ta)^{2\times 10^{-(S+1)}} R^{\frac{1}{3}(1-4\times 10^{(S+1)})} \prod_{K=1}^{S} (\log 1/g_K)^{\frac{1}{5}\times 10^{K-S}}, \tag{11}$$

where $\prod_{K=1}^{0} = 1$, S is the total number of R-modes, $\delta_n = g_n/\alpha_n$ is the thickness of the nth inner layer,

$$b_{n+1} = 10^{\frac{1}{3}n - \frac{1}{27}(1-10^{-n})} (\sigma/\beta)^{\frac{1}{3}(1-10^{-n})} \left[\frac{3}{2\sigma(8\times 10^S - 1)}\right]^{\frac{1}{3}[1-10^{-(n+1)}]}, \quad 0 \leqslant n \leqslant S-1, \tag{12}$$

$$K_S = (30\beta/I)^{\frac{6}{5}}\, 10^{\frac{4}{3}(S-1) - \frac{2}{135}(1-10^{-S+1})} (\sigma/\beta)^{\frac{4}{3}(1-10^{-S})} \left[\frac{3}{2\sigma(8\times 10^S - 1)}\right]^{\frac{4}{3}[1-10^{-(S+1)}]}, \tag{13}$$

$$I = 1{\cdot}1106, \quad \beta = 0{\cdot}4539, \quad \sigma = 0{\cdot}2775.$$

For a given Ta in the range (8), the present analysis assumes that

$$\delta_S \ll 1/\alpha_S \ll \ldots \ll \delta_n \ll 1/\alpha_n \ll \ldots \ll 1/\alpha_1 \ll \delta_E \ll Ta^{-\frac{1}{4}} \quad \text{as} \quad a_n \to \infty, \tag{14}$$

where α_n is the wavenumber for an R-mode. Using (8)–(10) and (14), we obtain the following results after neglecting the logarithmic terms.

(i) For Ta in the range (5), there are infinitely many R-modes and finitely many T-modes. However, as Ta approaches $O(R^{\frac{4}{3}})$, the number of R-modes rapidly decreases. When $Ta = O(R^{\frac{4}{3}})$, $S \to 0$ and there are infinitely many T-modes.

(ii) For sufficiently large R, S can be assumed to be large in the range (5), and we obtain the following forms for the optimal values of S and N:

$$S = \frac{1}{\log 10}\left\{\log\left[\frac{9}{20}\log (R^{\frac{4}{3}}/Ta)\right]\right\}, \tag{15}$$

$$N = 0{\cdot}1855 R^{\frac{1}{3}} \{\log Ta\, (\log R^{\frac{4}{3}}/Ta)^{-10/9}\}^r, \quad r = \frac{4}{9\log (R^{\frac{4}{3}}/Ta)}. \tag{16}$$

Thus, for a given R in the range (5), N increases as Ta increases.

(iii) For $R^{\frac{4}{3}} \ll Ta \ll (R\log R)^{\frac{4}{3}}$, there are infinitely many T-modes and no R-mode. However, the E-mode itself is sufficient to optimize N, and N has the same form as in the single-mode case [I, equation (3.56)].

2.3. *The case* $(R\log R)^{\frac{4}{3}} \ll Ta \ll R^{\frac{3}{2}}$

The effect of rigid boundaries is not important for this case. Thus there are no significant differences between the results for this case and those for free boundaries (III). We refer to each mode for this case as a T-mode.

Each T-mode has three regions: the interior, an intermediate layer of thickness $Ta^{\frac{1}{2}}/\alpha_n^3$ and an inner layer of thickness $\delta_n = g_n/\alpha_n$. The interior of each mode coincides with the inner layer of the previous mode. Coupling among the different modes occurs only between the nth and the $(n-1)$th modes in the $(n-1)$th boundary layer. In addition to the T-modes, there is an Ekman layer of thickness $(4/Ta)^{\frac{1}{4}}$. It is assumed that

$$Ta^{-\frac{1}{4}} \ll \delta_S \ll Ta^{\frac{1}{2}}/\alpha_S^3 \ll \ldots \ll \delta_1 \ll Ta^{\frac{1}{2}}/\alpha_1^3 \quad \text{as} \quad \alpha_n \to \infty, \tag{17}$$

where S is the number of T-modes. The assumption that the Ekman layer is thinner than the δ_S layer is needed to adjust the solution to the boundary conditions. By making the assumption (7) and using the formal multi-boundary-layer technique, we find

$$\alpha_n = b_n R^{(1-3\times 2^{-n-1})} Ta^{\frac{1}{2}(2^{1-n}-1)} \prod_{K=1}^{n-1} (\log R^2 g_K/Ta^{\frac{3}{2}})^{2^{K-n-1}}, \tag{18}$$

$$g_n(\log R^2 g_n/Ta^{\frac{3}{2}}) \prod_{K=1}^{n-1} (\log R^2 g_K/Ta^{\frac{3}{2}})^{2^{K-n-1}} = Ta^{\frac{3}{2}-2^{-n}} R^{-2+3\times 2^{-(n+1)}}, \tag{19}$$

$$N = K_S[R^{\frac{3}{2}} Ta^{-1} \log (R^{\frac{3}{2}}/Ta)]^{2(1-2^{-S})} (\tfrac{1}{2})^{2(S-1+2^{-S})}, \tag{20}$$

$$b_{n+1} = 2^{n-\frac{3}{2}(1-2^{-n})}(2^{S+2}-3)^{-1+3\times 2^{-n-2}}, \quad 0 \leqslant n \leqslant S-1, \tag{21}$$

$$K_S = (2/\pi^2)\, 2^{4S+3(-1+2^{-S+1})}\, (2^{S+2}-3)^{3\times 2^{-S}-4}. \tag{22}$$

For sufficiently large R, S can be assumed to be large, and we find the following optimal values for S and N:

$$S = \frac{1}{\log 2}[\log\log (R^{\frac{3}{2}}/Ta)], \tag{23}$$

$$N = 0{\cdot}0052 R^3/Ta^2. \tag{24}$$

Equation (24) is essentially equivalent to that given by Chan (III) for free boundaries. However, he obtained a different proportionality constant, partly because he ignored the contributions of the terms $(Ta^2R^{-3})^{2^{-S}}$ and

$$\int_0^\infty f^2\, d\zeta$$

in his expression for the heat flux.

3. Some remarks

(i) An important improvement in our present solutions over the single-mode solutions is that here the functional dependence of N on R and Ta is continuous (within a logarithmic term) throughout the region of R, Ta space.

(ii) Another important result of our present analysis is that rotation can increase the optimal N. For free boundaries (III), it was found for a given R that the optimal N cannot be increased as Ta increases.

(iii) If the effect of a rotational constraint is to stabilize the maximizing flow, then, the stronger the rotation, the more it will tend to suppress small scales of motion, and therefore we should have fewer modes. Here we find that there are infinitely many modes in the ranges (4) and (5), but it may be seen from (15) and (23) that as the rotation increases the number of modes decreases.

(iv) The expressions (16) and (24) for N (with logarithmic terms neglected) are consistent with the well-known dimensional argument that at large R the dimensional form of N should be independent of d.

(v) As is discussed in I, the single-mode upper bound of the present problem agrees qualitatively with the experimental finding of Rossby (1969). However, this bound is larger than the available data, and is somewhat larger than Rossby's observed values. Here we find that the single-mode upper bound is preferred in Rossby's experimental regime while multi-modal upper bounds give correctly the preferred bounds for sufficiently large R for a given Ta.

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